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LETTER TO THE EDITOR

An exact solution for electromagnetic waves in a strongly nonlinear media

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Abstract. Explicit wave solutions are found for the electromagnetic field that develops in a family of strongly nonlinear dielectric media. The time frequency of oscillation ω, is shown to be linear in the corresponding 'wavenumber' k, which allows for a standing wave in a finite system. The velocity of propagation along trajectories of constant field in spacetime coordinates is found to be proportional to a power of the amplitude of the field. For the particular oscillatory solutions, this results in wiggly, rather than straight, characteristic lines.

A nonlinear response of media under the application of an external field is very common in nature, and is usually encountered under the application of sufficiently strong external fields. A strongly nonlinear medium is defined as a medium where the nonlinearity appears as the leading mode of behaviour even when the applied field is weak. This is to distinguish from weakly nonlinear materials, where the nonlinearity consists of a small correction to a leading linear response. The latter case enjoys many analytic and numerical studies in the literature. Contrarily, with the exception of lasers, the response of strongly nonlinear systems is much less investigated. Here we study dielectric systems whose nonlinearity is of the form

$$\boldsymbol{D}(\boldsymbol{r}, t) = \varepsilon|\boldsymbol{E}(\boldsymbol{r}, t)/e|^{\beta}\boldsymbol{E}(\boldsymbol{r}, t) \tag{1a}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \mu\boldsymbol{H}(\boldsymbol{r}, t) \tag{1b}$$

where ε, μ, $\boldsymbol{E}(\boldsymbol{r}, t)$, $\boldsymbol{B}(\boldsymbol{r}, t)$, $\boldsymbol{D}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$ are, respectively, the dielectric permittivity, the magnetic permeability, the electric field, the magnetic inductance, the displacement field and the magnetic field. The rescaling scalar e has the same units as the electric field and is introduced to take care of the dimensions. Its magnitude is usually determined by the microscopic structure, but for simplicity it will be assumed unity and consequently omitted in the following. The particular power law relation between the field and the response has been shown to yield somewhat to exact analysis, and was therefore used to study the static properties of nonlinear conducting networks [1] and both homogeneous and disordered dielectrics [2, 3]. It has been shown that a unique solution to Maxwell's equations exists only for $\beta+1>0$ [2], while for $\beta+1<0$ metastable solutions were found [4].

Focusing here on the regime $\beta+1>0$, I discuss the time dependent solution within such strongly nonlinear dielectric media. I show that Maxwell's equations admit oscillatory solutions for $\boldsymbol{E}$ and $\boldsymbol{H}$, and derive their explicit forms in a finite medium.

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I find the temporal and spatial frequencies of the resultant waves and show that these frequencies are related *linearly*. I discuss the energy flow and give the explicit form for the energy density and Poynting's vector. Finally the velocity of propagation of a signal is addressed. It is shown that the characteristic lines in such a medium are not straight, but for oscillatory solutions may rather oscillate around the straight line in spacetime coordinates.

Assume a semi-infinite space $-\infty<x<\infty$, $-\infty<y<\infty$ and $0<z<\infty$, occupied by a non-conducting and nonlinear dielectric material, that is free of charges and satisfies the constitutive relations (1). A monochromatic electromagnetic wave†, propagating in the positive z-direction is incident on the x-y plane at $z=0^-$. The first aim is to find the coordinate- and time-dependence of the electromagnetic response that develops inside the nonlinear medium, and which satisfies Maxwell's relations

$$\nabla\times\boldsymbol{H}=\partial_t\boldsymbol{D}/c \tag{2a}$$

$$\nabla\times\boldsymbol{E}=-\partial_t\boldsymbol{B}/c. \tag{2b}$$

Without loss of generality one can align the x-y axes in the directions of the orthogonal incident fields, such that $\boldsymbol{E}_i(z=0^-,t)=E_i(0^-,t)\boldsymbol{x}$ and $\boldsymbol{B}_i(z=0^-,t)=B_i(0^-,t)\boldsymbol{y}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are unit vectors in the x and y directions. The boundary conditions at $z=0$ consist of continuity of the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$, so

$$E(0,t)=E_i(0,t)+E_r(0,t) \qquad H(0,t)=H_i(0,t)+H_r(0,t)$$

where the subscript r stands for the reflected wave. For $z<0$ it is a textbook result that

$$H_i(0^-,t)=(\varepsilon_0/\mu)^{1/2}E_i(0^-,t) \qquad H_r(0^-,t)=-(\varepsilon_0/\mu)^{1/2}E_r(0^-,t).$$

So at $z=0^+$ we have for the magnitude of the fields

$$\sqrt{(\varepsilon_0/\mu)}\,E(0^+,t)+H(0^+,t)=\sqrt{(\varepsilon_0/\mu)}\,E_i(0^-,t)+H_i(0^-,t).$$

The above defines unambiguously the matching of the fields at $z=0$. The matching of the transmitted wave at a boundary $z=W>0$ for a finite medium is similarly simple.

Inspecting Maxwell's relations it is easy to see that since the nonlinearity in (1) does not disturb the orientations of $\boldsymbol{E}$ and $\boldsymbol{B}$, then these fields must remain in their original directions and mutually perpendicular within the nonlinear medium for $z>0$. The directions of the response fields $\boldsymbol{D}$ and $\boldsymbol{H}$ follow those of $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively. One can now combine equations (2) and use (1) to eliminate the magnetic field, which gives an equation for the magnitude of $\boldsymbol{E}$

$$\partial_{zz}E=\nu_0^{-2}\partial_{tt}(|E|^\beta E) \qquad \text{where } \nu_0=c/\sqrt{\mu\varepsilon}. \tag{3}$$

To solve (3) I first assume that $E(z,t)$ is separable into a product of spatial and temporal independent functions

$$E(z,t)=R(z)T(t)^{1/(\beta+1)}. \tag{4}$$

This separation yields a class of solutions that is of importance for finite systems, and also gives insight into the nature of a propagating signal discussed below. Substituting the form (4) into (3) and dividing by $[R^{\beta+1}T^{1/(\beta+1)}]$ yields two independent equations for R and T:

$$\partial_{zz}R=-K|R|^\beta R \tag{5a}$$

$$\partial_{tt}T=-K\nu_0^2|T|^{-\beta/(\beta+1)}T \tag{5b}$$

† For the purpose of this discussion 'monochromatic' means a wave with a well defined frequency and wavelength, but whose form is not necessarily sinusoidal.

where K is a constant that relates the two functions. K is assumed positive in this discussion, but it is easy to show that for $K<0$ equations (5) give decreasing power law solutions. These parallel the familiar exponential decay in the linear case, and can be similarly interpreted. This issue is discussed elsewhere in more detail [5].

For the spatial function R one solves in a standard way: multiply both sides by $\partial_z R$ and integrate to obtain

$$(\partial_z R)^2 + A|R|^{\beta+2} = u_z \tag{6}$$

where u_z is a constant of integration and $A \equiv 2K/(\beta+2)$. Equation (6) can be regarded as an *energetic relation* describing a non-dissipative motion of a particle in a potential well. The first term on the LHS of (6) can then be identified as the kinetic energy and the second represents a potential term. For $\beta>0$ $(-1<\beta<0)$ the potential well is steeper (shallower) than the parabolic form of the familiar harmonic oscillator (corresponding to $\beta=0$). It is therefore clear that (6) *accommodates oscillatory solution*. Thus we first note that when the kinetic term vanishes, the potential term equals u_z, which immediately provides the amplitude of R,

$$a_R = (u_z/A)^{1/(\beta+2)}. \tag{7}$$

Correspondingly, the amplitude of $\partial_z R$ is $u_z^{1/2}$. Equation (6) can be solved for z in the form of an indefinite integral over R

$$u_z^{1/2}(z-z_0) = \int^R \frac{dR'}{\pm[1-(A/u_z)|R'|^{\beta+2}]^{1/2}} \tag{8}$$

where z_0 is a constant of integration to be determined by the boundary conditions. Changing variables to $\zeta = (A/u_z)|R|^{\beta+2}$ one gets

$$(A/u_z)^{1/(\beta+2)} u_z^{1/2}(z-z_0) = \frac{\pm 1}{\beta+2}\mathfrak{B}_\zeta\left[\frac{1}{\beta+2}, \frac{1}{2}\right] \tag{9}$$

where $\mathfrak{B}_\zeta(a,b)$ $(0\leqslant\zeta\leqslant 1)$ is the incomplete beta function (see e.g. [6]). It is easy to show by expanding the integrand in (8) that near the origin $(z-z_0=0)$ R is linear in $(z-z_0)$, while near the maximum in R, $R_{\max}-R \sim |z-z_{\max}|^2$. These behaviours are independent of β and generalize the linear case, when R is sinusoidal. $R(z)$ is shown in figure 1 for $\beta=-1/2$ and 1, in the first quarter of the period. The rest of the period is facilitated by the possibility of both signs in the integral (8), combined with the possibility of both signs for R' in the denominator in (8). The entire period of oscillation is $[4/(\beta+2)]\mathfrak{B}[1/(\beta+2), 1/2]$, where $\mathfrak{B}(a,b) = \mathfrak{B}_{\zeta=1}(a,b)$ is the usual beta function

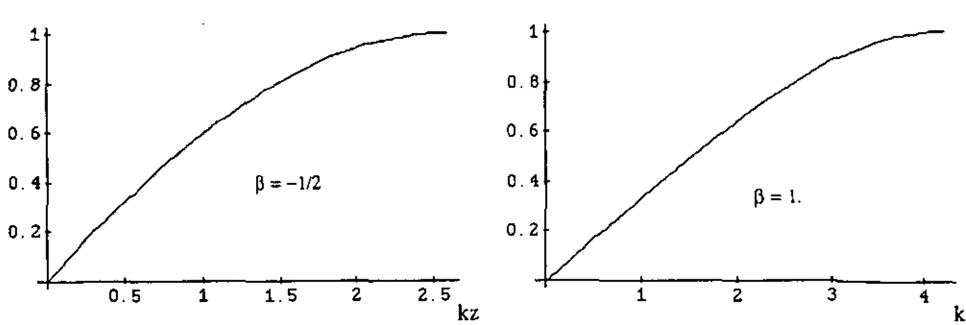

Figure 1. The solutions for $(A/u_z)^{1/(\beta+2)}R$, for $\beta=-1/2$ and $\beta=1$. The values of k on the abscissa are scaled such that the period is $[4/(\beta+2)]\mathfrak{B}[1/(\beta+2), 1/2]$.

[6]. This period reduces to 2π as expected for $\beta=0$. For later use, let us also identify the analogue of the 'wavenumber', defined such that the period is unity (not 2π)

$$k=(A/u_z)^{1/(\beta+2)}\frac{(\beta+2)u_z^{1/2}}{4\mathfrak{B}[1/(\beta+2),1/2]}. \tag{10}$$

Now let us turn to the temporal behaviour. The differential equation (5*b*) can be solved in the same manner as (5*a*) through multiplication by $\partial_t T$ and integration, which yields

$$(\partial_t T)^2+A_1|T|^{(\beta+2)/(\beta+1)}=u_t \tag{11}$$

where u_t is a constant of integration and $A_1\equiv A\nu_0^2(\beta+1)$. Very similarly to equation (6), for all $-1<\beta$ this relation describes a non-dissipative oscillatory behaviour in a generally non-quadratic potential well (except for $\beta=0$). The amplitude of T can be found by considering the instant when $\partial_t T=0$,

$$a_T=(u_t/A_1)^{(\beta+1)/(\beta+2)}. \tag{12}$$

From equations (4), (7) and (12) we can now deduce the total amplitude of the electric field

$$E_0=a_R a_T^{1/(\beta+1)}=\left[\left(\frac{\beta+2}{2K}\right)^2\frac{u_z u_t}{\nu_0^2(\beta+1)}\right]^{1/(\beta+2)}. \tag{13}$$

Exactly as above, equation (11) can be inverted to solve for the time t

$$(A_1/u_t)^{(\beta+1)/(\beta+2)}\frac{\beta+2}{\beta+1}u_t(t-t_0)=\pm\mathfrak{B}_\eta\left(\frac{\beta+1}{\beta+2},\frac{1}{2}\right) \tag{14}$$

where $\eta=(A_1/u_t)|T|^{(\beta+1)/(\beta+2)}$, and t_0 is some initial time. When T is expressed in terms of t, relation (14) yields the expected oscillatory behaviour with a period of $[4(\beta+1)/(\beta+2)]\mathfrak{B}((\beta+1)/(\beta+2),1/2)$. As expected this period also reduces to 2π when $\beta=0$. The frequency of this oscillation can be deduced from (14)

$$\omega=(A_1/u_t)^{(\beta+1)/(\beta+2)}u_t^{1/2}\Big/\mathfrak{B}\left(\frac{\beta+1}{\beta+2},\frac{1}{2}\right). \tag{15}$$

In the linear case the ratio ω/k gives the dispersion relation and the phase velocity so it is of interest to consider this ratio here. Using equations (10) and (15) yields

$$\omega/k=E_0^{-\beta/2}\sqrt{\beta+1}\,\nu_0\mathfrak{B}\left(\frac{1}{\beta+2},\frac{1}{2}\right)\Big/\mathfrak{B}\left(\frac{\beta+1}{\beta+2},\frac{1}{2}\right) \tag{16}$$

which reduces to the usual ν_0 in the linear case as it should. Expression (16) yields a spectacular interpretation: since E_0 and β depend neither on time nor on spatial coordinates ω is *linear* in k.

The first implication of this intriguing result is that despite being strongly nonlinear, a non-dissipative medium can sustain a persistent standing wave. Namely, if one pumps a wave of some frequency ω at $z=0$ onto a medium, confined between $z=0$ and $z=W<\infty$, the result is a standing wave. This standing wave will assume definite periodicities that are integer multiples of the basic period, just as in the linear case. But unlike the linear case a simple linear combination of these solutions is not a solution also because the superposition principle does not apply. The second ramification relates to the intensity dependence. Relation (16) shows that increasing the amplitude E_0 of the incident wave at $z=0$ decreases (increases) the phase velocity for $\beta>0$ ($-1<\beta<0$). Thus one can modulate the wavenumber k of such a standing wave by varying the *amplitude* E_0 rather than the frequency ω of the source. It is not unusual to find that nonlinearity couples frequency and amplitude. However, here we have an exact relation between the two, rather than a truncation of some expansion in a small parameter.

So the electric field inside the nonlinear medium has been found to oscillate both in space and time, with the oscillating functional form being the inverse of the incomplete beta function with respect to its index. Regarding the boundary conditions, we note that for the above calculation to remain valid first the incident wave at $z=z_0$ has to be prearranged such that it follows the same form as $R(z=z_0)T(t=t_0)^{1/(\beta+1)}$. If this is not taken care of one may end up with a mismatch at the boundary leading to a possible dispersion at $z=0^+$. Since the superposition principle cannot be applied in our medium, such a situation complicates the calculation drastically. Secondly, the reflected amplitude at z_0 should be equal to the incident.

Next let us consider the behaviour of the magnetic field $\boldsymbol{H}(z,t)$ within the nonlinear medium. Rewriting Maxwell's equations (2) in terms of the functions R and T, integrating and using equations (5), (6) and (11) H can be found explicitly

$$H=\pm\frac{\varepsilon}{cK}\partial_z R\,\partial_t T=\pm\frac{\varepsilon(u_z u_t)^{1/2}}{cK}\sqrt{[1-(A/u_z)|R|^{\beta+2}][1-(A_1/u_t)|T|^{(\beta+2)/(\beta+1)}]}. \qquad (17)$$

The expressions within the square roots are the canonical wave solutions to the energetic equations (6) and (11), which vary with z and t between 0 and 1. The amplitude of H is then the prefactor, which can be identified, using (13), as

$$H_0=\frac{2}{\beta+2}\sqrt{\frac{\varepsilon(\beta+1)}{\mu}}\,E_0^{(\beta+2)/2}. \qquad (18)$$

Having found the explicit forms of the electric and the magnetic fields, I now proceed to discuss the energy stored in the electromagnetic field and its flow in the nonlinear medium. Consider Poynting's vector defined by

$$\boldsymbol{S}=(c/4\pi)\boldsymbol{E}\times\boldsymbol{H}. \qquad (19)$$

The divergence of $\boldsymbol{S}$ can be easily calculated in any general medium (see e.g. [7])

$$\operatorname{div}\boldsymbol{S}+(\boldsymbol{H}\circ\partial_t\boldsymbol{B}+\boldsymbol{E}\circ\partial_t\boldsymbol{D})/4\pi=0$$

and in the case of the nonlinear constitutive relations (1), can be written as

$$\operatorname{div}\boldsymbol{S}+\partial_t U=0 \qquad (20)$$

where

$$U=\frac{1}{4\pi}\left(\frac{\beta+1}{\beta+2}\varepsilon|E|^{\beta+2}+\frac{\mu}{2}|H|^2\right) \qquad (21)$$

is *exactly the energy density* in the system. It should be emphasized that although one can write equation (20) for any nonlinear system, U need not, and generally does not, coincide with the energy density in the system. Thus within a period of oscillation the energy density is exchanged between the magnetic and electric fields. However, the presence of the non-unity factor in front of the magnetic energy indicates that when H is maximal (and E vanishes) the energy density is different (smaller for $\beta>0$ and larger for $0>\beta>-1$) than when E is maximal (and H vanishes).

It should be stressed that by assuming a separable solution, the above discussion focused implicitly on the case of a standing, rather than a propagating, wave. I now turn to consider the propagation of a signal in such a medium. In the linear case the fields E and H can be written as functions of the reduced variable $x = z \pm \nu_0 t$, which shows that a signal of a well defined unique frequency will propagate in the speed of light ν_0 both forward and backward in the corresponding medium (complications arising when several modes propagate [8] are avoided here by considering only one frequency). The question is: can one identify a quantity that is the analogue of the propagation velocity of the signal in our case? Let us assume that there exists such a velocity ν_β. The field E can be written then as a function of the reduced variable $\xi = (t - t_0) - (z - z_0)/\nu_\beta$ (to simplify the notation only forward propagation is considered). The first partial derivatives of E can be rewritten as

$$\begin{aligned} \partial_z E &= -\frac{E'}{\nu_\beta[1 + (z - z_0) d_E(1/\nu_\beta) E']} \\ \partial_t E &= \frac{E'}{1 + (z - z_0) d_E(1/\nu_\beta) E'} \end{aligned} \tag{22}$$

where $d_E = \mathrm{d}/\mathrm{d}E$ represents derivative with respect to the explicit dependence on E and where E' is the derivative of E with respect to the reduced variable ξ. Assuming that $[1 + (z - z_0) d_E(1/\nu_\beta) E']$ does not vanish we have

$$\partial_z E = -(1/\nu_\beta)\partial_t E$$

which, combined with the identity $\partial_z E/\partial_t E = -1/(\partial z/\partial t)_E$, leads to

$$(\partial z/\partial t)_E = \nu_\beta. \tag{23}$$

It follows that ν_β is the velocity along the trajectories of constant field E in the z-t plane.

I now claim that

$$\nu_\beta = \nu_0 |E|^{-\beta/2}/\sqrt{\beta + 1} \tag{24}$$

and proceed to prove it by showing that this expression is self-consistent and solves Maxwell's equations. Identifying $1/\nu_\beta^2 = (\mu/c^2) d_E D$ and using equation (2*a*) one has

$$\partial_z H = -(c/\mu)(1/\nu_\beta^2)\partial_t E. \tag{25}$$

Using (23) in (25) and changing variables yields

$$H = (c/\mu) \int \mathrm{d}E/\nu_\beta$$

which can be easily checked by differentiation with respect to time and comparing with equation (2*b*). Explicitly, this relation yields for the magnitude of H

$$H = \frac{2\sqrt{(\beta + 1)\varepsilon/\mu}}{\beta + 2} |E|^{\beta/2} E \tag{26}$$

which coincides with equations (17) and (18) up to a phase shift. Expression (26) also shows that in the case of a propagating signal, H is *in phase* with E (rather than in antiphase as in the case of a standing wave), exactly as in the linear case.

Further, this calculation shows that *any function* of ξ solves Maxwell's equations for E and H. This generalizes the linear case result, where any function of $z - \nu_0 t$ is a solution, depending on the initial conditions.

So by writing E in terms of ξ the problem is reduced from being described by two variables (z, t) to just one. This reduction may seem cumbersome due to the dependence of ξ on E through ν_β, but it is still useful as it provides insight when analysing the stability of the form of a propagating signal in such a medium. This issue is beyond the scope of the present paper and is discussed elsewhere [5]. I will only remark here that a signal propagating with a velocity that follows (24) may evolve into a frontal or rear shock-wave-like form.

The mean velocity of propagation $\langle\nu_\beta\rangle$, can be found in two ways: one is by averaging (25) directly over z and t, which gives $\langle\nu_\beta\rangle = \nu_0\langle E^{-\beta/2}\rangle/(\beta+1)^{1/2} \sim E_0^{-\beta/2}$. Another way is to consider the propagation of the energy flux through the media. Relation (20) (which is general, as mentioned above) has the form of a continuity equation and, when averaged over z and t, constitutes the conservation of energy in the system. The velocity of propagation is then simply $\langle S\rangle/\langle U\rangle$. Using S and U for the previous case of separable solution yields

$$\langle\nu_\beta\rangle = \frac{2(\beta+2)}{(\beta+4)(\beta+1)^{1/2}}\nu_0\frac{\langle E^{\beta/2+1}\rangle}{\langle E^{\beta+2}\rangle} \sim E_0^{-\beta/2}$$

which varies with the same power of the field amplitude as in the first method. As expected, this power vanishes in the linear case, leading to the familiar field-independent constant velocity.

Thus, ν_β indeed represents a local and instantaneous velocity of propagation of the solution inside the nonlinear medium, and it varies with the spatial and temporal coordinates, tracing the variation of the field itself. Information can be carried by this solution, e.g. by introducing a perturbation at some t and z. Although, unlike in the linear case, the propagation of a general perturbation is difficult to analyse, this perturbation will distort the field, and such a distortion can be detected at another location and later time. This information propagates with the average velocity, and hence the instantaneous velocity is not the most important quantity to the passage of information over extended distances.

To conclude, I have analysed the electromagnetic response of a strongly nonlinear dielectric medium. I have presented exact wave solutions to Maxwell's equations, whose functional form has been derived explicitly. The temporal frequency of oscillation ω has been shown to vary linearly with the analogue of the wavenumber k. This indicates that such a nonlinear medium can sustain a standing wave. The ratio ω/k (phase velocity) within the nonlinear medium has been shown to depend on the intensity of the incoming wave through a power law form, which allows one to modulate the wavelength inside the medium by varying the intensity of the source wave, rather than varying its frequency. The energy flux and Poynting's vector have been discussed for these solutions. The velocity derivative $\partial z/\partial t$ along trajectories of constant field in the z–t plane has been found to be a simple power of the field intensity for any general solution. In particular, when the field oscillates this velocity traces this oscillation and consequently the characteristic lines of constant field may also oscillate periodically around the straight line. Some points remain unclear in the propagating case regarding the stable form of the signal. The simplicity in the linear case stems from the easy decomposition of the plane wave in $x = z - \nu_0 t$ into a sum of two separable periodic functions in kz and ωt. Such a decomposition is not available to us in the present nonlinear problem, and this question is currently under study.

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